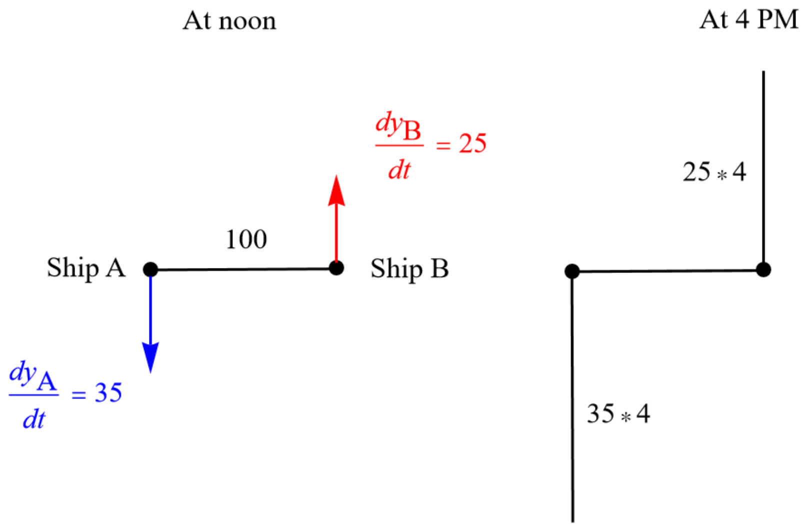
Exercise 23

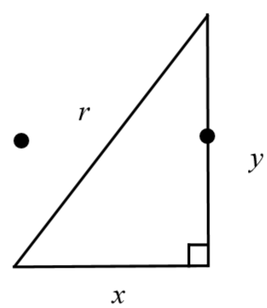
At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?

Solution

Draw a schematic of the two ships' paths. The length of each path is obtained by multiplying a ship's speed by the time travelled.



Let the distance between the ships at any time be r.



The sides of this triangle are related by the Pythagorean theorem.

$$r^2 = x^2 + y^2$$

Take the derivative of both sides with respect to t and use the chain rule.

$$\frac{d}{dt}(r^2) = \frac{d}{dt}(x^2 + y^2)$$

$$2r \cdot \frac{dr}{dt} = 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}$$

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

Solve for dr/dt.

$$\frac{dr}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{r}$$

$$= \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2 + y^2}}$$

Use the fact that y is the sum of the ships' distances: $y = y_A + y_B$.

$$\frac{dr}{dt} = \frac{x\frac{dx}{dt} + (y_A + y_B)\frac{d}{dt}(y_A + y_B)}{\sqrt{x^2 + (y_A + y_B)^2}}$$

$$= \frac{x\frac{dx}{dt} + (y_A + y_B)\left(\frac{dy_A}{dt} + \frac{dy_B}{dt}\right)}{\sqrt{x^2 + (y_A + y_B)^2}}$$

$$= \frac{x\frac{dx}{dt} + (y_A + y_B)(35 + 25)}{\sqrt{x^2 + (y_A + y_B)^2}}$$

$$= \frac{x\frac{dx}{dt} + 60(y_A + y_B)}{\sqrt{x^2 + (y_A + y_B)^2}}$$

The horizontal distance between the ships doesn't increase as time passes, so $dx/dt = 0$.

$$\frac{dr}{dt} = \frac{x(0) + 60(y_A + y_B)}{\sqrt{x^2 + (y_A + y_B)^2}}$$

$$= \frac{60(y_A + y_B)}{\sqrt{x^2 + (y_A + y_B)^2}}$$

Therefore, at 4:00 PM, the rate the distance between the ships is changing is

$$\left.\frac{dr}{dt}\right|_{t=4} = \frac{60(35 * 4 + 25 * 4)}{\sqrt{(100)^2 + (35 * 4 + 25 * 4)^2}} = \frac{720}{13} \approx 55.3846 \ \frac{\text{km}}{\text{h}}.$$